The Schrodinger equation for the f(x)/g(x) interaction

1990 J. Phys. A: Math. Gen. 23 L1109

(http://iopscience.iop.org/0305-4470/23/21/007)

J. Phys. A: Math. Gen. 23 (1990) L1109–L1112. Printed in the UK

LETTER TO THE EDITOR

The Schrödinger equation for the $f(x)/g(x)$ interaction

D Hislop, M F Wolfaardt and P G L Leach†

CAMS-CNLS, University of the Witwatersrand, PO WITS-2050, South Africa

Received 4 June 1990

Abstract. A natural extension of the $\omega^2x^2+\lambda x^2/(1+gx^2)$ interaction is $\omega^2x^2+f(x)/g(x)$ where f and g are polynomials in x^2. The method of obtaining quasi-exact solutions of the Schrödinger equation is outlined and some simple examples given. Solutions which are valid for part of the real line are obtained when $g(x)$ has real zeros.

The study of the quasi-exactly solvable Schrödinger equation with model potentials is by now well established. Earlier papers (Flessas 1979, 1981a, b, Flessas and Das 1980, Khare 1981, Magyari 1981) tended to be a little on the *ad hoc* side. A more systematic approach was initiated by Leach (1984, 1985). The next logical development was the use of symbolic manipulation codes such as REDUCE to do the algebra (Blecher and Leach 1987a, b). (Note that this need not imply that the final results are necessarily correct; see the comment by Gallas (1988).) Two problems have usually been studied because of their common occurrence in areas of physical interest—the polynomial anharmonic oscillator and the $x^2+\lambda x^2/(1+gx^2)$ interaction in both one and three dimensions. Recently the screened Coulomb potential has been added to the list (Maccelari and Leach 1989). An exhaustive study of polynomial anharmonic oscillator potentials in one dimension for which the Schrödinger equation is quasi-exactly solvable was made by Leach *et al* (1989). Some aspects of this work were further developed by Pursey (1990). Some aspects of non-separable polynomials potentials in two dimensions were reported by Taylor and Leach (1989). The general approach to these problems is to make an ansatz of the structure of the wavefunction in the form of polynomial multiplied by a suitable exponential to ensure it vanishes at infinity. (All of the problems referred to above have been on R^1, R^2 or $(0,\infty)$.) This procedure is by now well established and further expositions would probably be pleonastic. In this letter we wish to follow a different tack and explore the natural generalizations of the $x^2+\lambda x^2/(1+gx^2)$ interaction. Thus we look at potentials of the form $f(x)/g(x)$ where $f(x)$ and $g(x)$ are polynomials of even order. We do not attempt an exhaustive, but rather indicate the variety of possibilities for which quasi-exact solutions occur.

We take as our model Hamiltonian

$$H=\tfrac{1}{2}(p^2+\omega^2x^2+f(x)/g(x)) \tag{1}$$

where $f(x)$ and $g(x)$ are polynomials in x^2 of degree N. (In practice the degree of $f(x)$ may be less than N.) The corresponding time-independent Schrödinger equation is

$$\phi''+(\lambda-\omega^2x^2-f(x)/g(x))\phi=0. \tag{2}$$

† Present address: Department of Mathematics and Applied Mathematics, University of Natal, King George V Avenue, Durban 4001, South Africa.

0305-4470/90/211109+04\$03.50

Following the observation made by Blecher and Leach (1987b) we make the substitution

$$\phi(x)=g(x)y(x) \tag{3}$$

to balance the equation at the poles of $g(x)$. Then $y(x)$ satisfies

$$gy''+2g'y'+((\lambda-\omega^2x^2)g-f+g'')y=0. \tag{4}$$

In the case that $g(x)$ has no real zeros, (4) can be applied to $(-\infty,\infty)$ with $y(\pm\infty)=0$ as boundary conditions. The asymptotic behaviour of y which satisfies these boundary conditions is

$$y\sim\exp(-\tfrac{1}{2}\omega x^2) \tag{5}$$

and, if we make the substitution

$$y(x)=u(x)\exp(-\tfrac{1}{2}\omega x^2) \tag{6}$$

$u(x)$ is a solution of

$$gu''+2(g'-\omega xg)u'+\{(\lambda-\omega)g-f-2\omega xg'+g''\}u=0. \tag{7}$$

Given that

$$f(x)=\sum_0^N a_ix^{2i} \qquad g(x)=\sum_0^N b_ix^{2i} \tag{8}$$

we can ensure that wavefunction is square integrable by making the ansatz that

$$u(x)=\sum_0^M k_ix^{2i+\varepsilon} \tag{9}$$

where $\varepsilon=0$ for an even solution and $\varepsilon=1$ for an odd solution. When (2.8) and (2.9) are substituted into (2.7) and coefficients of like powers of x are collected and equated to zero, we obtain a system of constraints on the coefficients a_i, b_i and k_i and the eigenvalue λ. There $M+N+1$ constraints for the $M+2N+2$ parameters. (The number of parameters is not $M+2N+4$ since in the ratio f/g a degree of freedom is lost and another is lost in the normalization of the wavefunction. The easiest way to reduce the ambiguity is to set $b_N=1$.) We illustrate the solution process with a simple example below.

In previous studies $g(x)$ has been the quadratic $1+gx^2$ which does not have real zeros. However, there is no reason for $g(x)$ not to have real zeros. If $g(x)$ becomes zero for real x (and this would be pairwise since $g(x)$ is a polynomial in x^2), (2.4) is not necessarily an eigenvalue problem over $(-\infty,\infty)$. If the zeros are simple, infinite barriers are created in the potential and the eigenvalue problem is confined to sections of the real line between singularities or between a singularity and $+\infty$ or $-\infty$. If $\omega=0$, the latter possibility cannot occur. The process of finding a wavefunction is the same as outlined above for $g(x)$ not having real zeros with one exception. For a solution between two singularities, it is not *necessary* to include the exponential term. Simple example are considered below.

We take $f(x)$ and $g(x)$ to be quartic, namely

$$f(x)=a_0+a_1x^2+a_2x^4 \tag{10}$$

$$g(x)=b_0+b_1x^2+x^4 \qquad 4b_0-b_1^2>0 \tag{11}$$

where, as noted above, b_2 is set at unity to avoid any ambiguity. To keep the algebra to a minimum we take

$$u(x) = k_0 + x^2 \tag{12}$$

where k_1 is set at unity to allow for an overall normalization factor. After (10)-(12) are substituted into (7) and coefficients of independent powers of x are put equal to zero, we find that

$$\lambda = a_2 + 13\omega \tag{13}$$

$$k_0 = \frac{1}{4\omega}[a_1 - 30 - (a_2 + 4\omega)b_1] \tag{14}$$

$$[(a_2 + 8\omega)b_1 + 12 - a_1]k_0 + 8b_1 - a_0 + (a_2 + 8\omega)b_0 = 0 \tag{15}$$

$$2b_0 + [(a_2 + 12\omega)b_0 - a_0 - 2b_1]k_0 = 0. \tag{16}$$

Equations (15) and (16) are linear in a_0 and b_0 and can be solved to give a_0 and b_0 in terms of a_1, a_2, b_1 and ω. Assuming that ω is fixed there is a three-parameter family of potentials. The parameters are constrained by the requirements that $4b_0 - b_1^2 > 0$. The wavefunction will be either the ground state ($k_0 > 0$) or the second excited state ($k_0 < 0$). It is interesting to note that the eigenvalue is independent of the values of a_1 and b_1 which, however, do affect the shape of the wavefunction.

This situation is simply represented by the choices of f, g and u to be

$$f = a_0 \tag{17}$$

$$g = b^2 - x^2 \tag{18}$$

$$u = k + x^2 \tag{19}$$

which, when substituted into (7) yield

$$\lambda = \omega \tag{20}$$

$$k = b^2 + (12 + a_0)/(4\omega) \tag{21}$$

$$a_0 = (7 + 2\omega b^2) \pm \sqrt{[(5 + 2\omega b^2)^2 + 8\omega b^2]}. \tag{22}$$

With the + sign in (22), $k > 0$ and with the − sign, $k < 0$. The wavefunction $\phi(x)$ applies separately to the three intervals, $(-\infty, -b)$, $(-b, b)$ and (b, ∞). In the two semi-infinite intervals the wavefunction represents the ground state no matter the sign of k. However, in $(-b, b)$, the wavefunction represents the ground state for $k > 0$ and the second excited state for $k < 0$. Where $\omega = 0$, the solution would apply to $(-b, b)$ only. Again the wavefunction would represent the ground state or second excited state depending upon the sign of k. We note that the two solutions correspond to different potentials.

In this brief letter we have pointed out that a variety of potentials which are a natural generalization of the $\omega^2x^2 + \lambda x^2/(1 + gx^2)$ interaction is quasi-exactly solvable. In particular we saw that a denominator with real zeros lies within this class.

This work was supported in part by the Foundation for Research Development of South Africa.

References

Blecher M H and Leach P G L 1987a *Proc. 13th South African Symp. on Numerical Mathematics* ed P J Vermeulen (Durban: University of Natal) pp 5-17

—— 1987b *J. Phys. A: Math. Gen.* **20** 5923-7

Flessas G P 1979 *Phys. Lett.* **27A** 289-90

—— 1981a *Phys. Lett.* **81A** 17-8

—— 1981b *J. Phys. A: Math. Gen.* **14** L209-11

Flessas G P and Das K P 1980 *Phys. Lett.* **78A** 19-21

Gallas Jason A C 1988 *J. Phys. A: Math. Gen.* **21** 3393-7

Khare A 1981 *Phys. Lett.* **83A** 237-8

Leach P G L 1984 *J. Math. Phys.* **25** 2974-8

—— 1985 *Physica* **17D** 331-8

Leach P G L, Flessas G P and Gorringe V M 1989 *J. Math. Phys.* **30** 406-12

Maccelari J C and Leach P G L 1989 Closed form solutions of the Schrödinger equation for a screened Coulomb potential *Preprint CNLS-89-104* University of the Witwatersrand

Magyari E 1981 *Phys. Lett.* **81A** 116-8

Pursey D L 1990 Properties of Leach-Flesses-Gorringe polynomials *Preprint* Department of Physics, Iowa State University, Ames, Iowa, 50011

Taylor D R and Leach P G L 1989 *J. Math. Phys.* **30** 1525-1532